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MOTION OF SYSTEM OF PARTICLES AND RIGID BODY

Unit: - 5

CLASS: - 11TH

PHYSICS

Education source

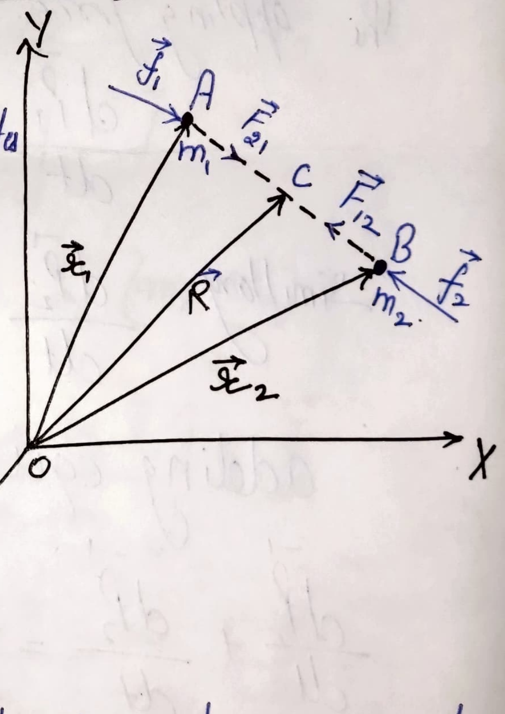
Systems of Particles and Rotational Motion ①

Unit :- 5

Centre of Mass and Rotational Motion :-

Centre of Mass :- A Point which the entire mass of the body or system of bodies is supposed to be Concentrated is known as the Centre of mass.

Centre of Mass of a two Particle System:-



Let us Consider two particle System in rectangular Co-ordinates XYZ.

Here $\vec{OA} = \vec{r_1}$, $\vec{OB} = \vec{r_2}$ and $\vec{OC} = \vec{R}$ → Position vectors

To evaluate R :-

V_1, V_2 and m_1, m_2 be the velocities and masses of the objects at Point A and B at any instant (t)

$$\vec{V_1} = \frac{d\vec{r_1}}{dt} \quad \text{and} \quad \vec{V_2} = \frac{d\vec{r_2}}{dt} \qquad \text{—— ①}$$

Let $\vec{f}_1$ and $\vec{f}_2$ be the external forces on mass m_1 and m_2

and $\vec{F}_{12}$ and $\vec{F}_{21}$ be the Internal forces on each other

$\vec{F}_{12}$ = force on m_1 due to m_2

and $\vec{F}_{21}$ = force on m_2 due to m_1

linear momentum of mass m_1 $\vec{P}_1 = m_1 \vec{v}_1$ — (II)

Acc. to Newton's second law

The linear momentum in a body due to the appling force. $(\vec{f}_1 + \vec{F}_{12})$

$$\frac{d\vec{P}_1}{dt} = \vec{f}_1 + \vec{F}_{12} \quad \text{— (III)}$$

Simillary
$$\frac{d\vec{P}_2}{dt} = \vec{f}_2 + \vec{F}_{21} \quad \text{— (IV)}$$

adding equation (III) and (IV)

$$\frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} = \vec{f}_1 + \vec{F}_{12} + \vec{f}_2 + \vec{F}_{21} \quad \text{— (V)}$$

Here Newton's third law

$$\vec{F}_{12} = -\vec{F}_{21} \Rightarrow \vec{F}_{12} + \vec{F}_{21} = 0$$

equation (V) becomes

$$\frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} = \vec{f}_1 + \vec{f}_2 = \vec{f} \quad \text{— (VI)}$$

$\vec{f}$ = Total force on the System.

Here from equation (VI) and (II)

$$\frac{d(m_1\vec{v}_1)}{dt} + \frac{d(m_2\vec{v}_2)}{dt} = \vec{f}$$

$$\frac{d}{dt}\left[(m_1\vec{v}_1 + m_2\vec{v}_2)\right] = \vec{f}$$

$$\frac{d}{dt}\left[m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt}\right] = \vec{f} \quad \text{[from equation I]}$$

$$\frac{d}{dt}\left[\frac{d}{dt}\left[m_1\vec{r}_1 + m_2\vec{r}_2\right]\right] = \vec{f}$$

$$\frac{d^2}{dt^2}\left[m_1\vec{r}_1 + m_2\vec{r}_2\right] = \vec{f}$$

multiplying and dividing by $(m_1 + m_2)$

$$(m_1 + m_2)\frac{d^2}{dt^2}\left[\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\right] = \vec{f}$$

(VII)

Here $\left[\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\right] \rightarrow$ total mass at point (C)

$$\Rightarrow (m_1 + m_2)\frac{d^2(\vec{R})}{dt^2} = \vec{f}\Bigg] \rightarrow$$ this is equation of motion.

Equation (VII) becomes

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\boxed{(m_1 + m_2)\vec{R} = m_1 \vec{r}_1 + m_2 \vec{r}_2} \quad \text{— (VIII)}$$

Hence Position vector of Centre of mass of a two particle system is the product of total mass and Position vector of Centre of mass is equal to sum of the products of masses of the two particles and their respective Position vectors.

Cases:-

I) Centre of mass of two particles of the system were at the origin

ie $\vec{R} = 0$

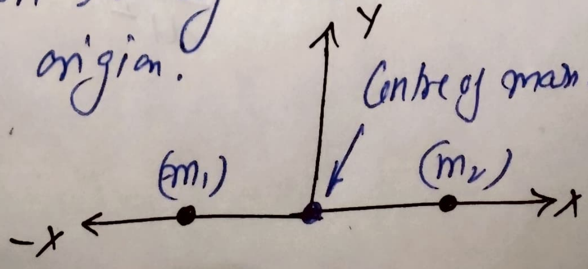
equation (VII) becomes

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

$$\boxed{\vec{r}_2 = -\frac{m_1 \vec{r}_1}{m_2}}$$

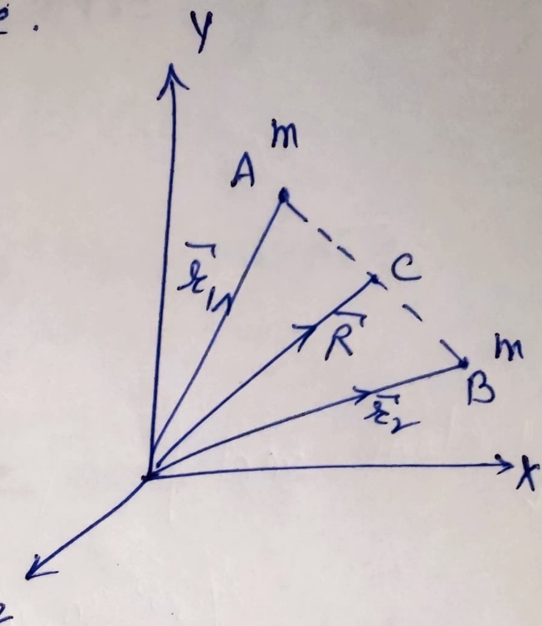
Show that $\vec{r}_1$ is (-ve) and $\vec{r}_2$ is (+ve)

that means m_1 on the left and m_2 lie on the Right of the origin.



② If $m_1 > m_2$ the $\vec{r}_1 < \vec{r}_2$

ie Centre of mass of two particle system lies closer to the heavier particle.



③ if $m_1 = m_2 = m$

$$(m+m)\vec{R} = m\vec{r}_1 + m\vec{r}_2$$

$$2\cancel{m}\vec{R} = \cancel{m}(\vec{r}_1 + \vec{r}_2)$$

$$\boxed{\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}}$$

Centre of mass of two particles of equal masses lies exactly midway between the line joining

⇒ Momentum Conservation and Centre of Mass Motion :-

We know that Position vector of two particle system

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

Simillary Position vector for n-particle system.

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{M} \quad \text{—(I)}$$

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n$$

differentiating both side

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \cdots + m_n\frac{d\vec{r}_n}{dt}$$

we know that

$$\vec{V} = \frac{d\vec{R}}{dt}, \quad \vec{V}_1 = \frac{d\vec{r}_1}{dt} \cdots \vec{V}_n = \frac{d\vec{r}_n}{dt}$$

$$\left[M\vec{V} = m_1\vec{V}_1 + m_2\vec{V}_2 + \cdots + m_n\vec{V}_n\right] \quad \text{—(II)}$$

Here this equation shows that momentum Conservation.

$$\boxed{\vec{V} = \frac{m_1\vec{V}_1 + m_2\vec{V}_2 + \cdots + m_n\vec{V}_n}{M}}$$

differentiating both side equation (II)

$$M\frac{d\vec{V}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + \dots + m_n\frac{d\vec{v}_n}{dt}$$

We know that

acceleration $\vec{A} = \frac{d\vec{V}}{dt}$, $\vec{a}_1 = \frac{d\vec{v}_1}{dt}$, $\vec{a}_2 = \frac{d\vec{v}_2}{dt}$ &, --- $\vec{a}_n = \frac{d\vec{v}_n}{dt}$

$$M\vec{A} = m_1\vec{a}_1 + m_2\vec{a}_2 + \dots + m_n\vec{a}_n$$

Acc. to Newton's Second law

$$\vec{F} = m\vec{a}$$

ie $\vec{F} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n$

$$\boxed{F_{ext} = M\vec{A}} \quad \text{--- (III)}$$

all the internal forces cancel out each other only external forces can contribute in the motion of the particle System.

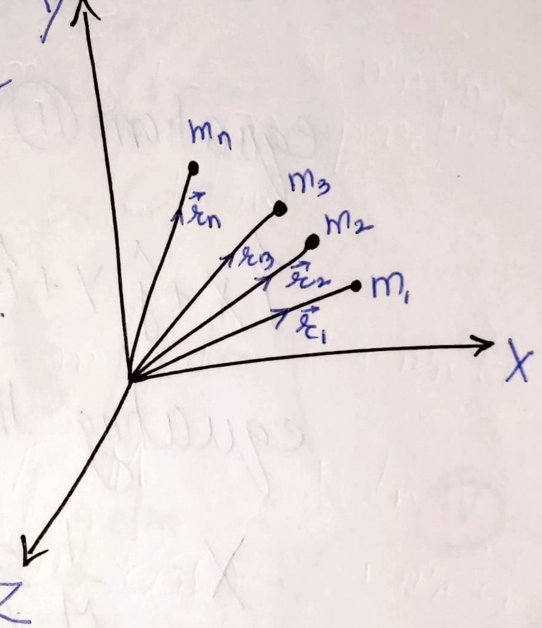
Equation of (III) Represents the Motion of Centre of Mass.

⇒ Centre of mass of Rigid body :-

A body is said to be rigid body when it has a perfectly definite shape and size.

eg:- A wheel can be considered as rigid body.

Let us consider n-particles in a rigid body



if $(\vec{r}_i)$ be the position vector of ith particle and (m_i) be the mass of that particle.

Here

$$\vec{R} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} \quad \text{--- (I)}$$

Here $\sum_{i=1}^{n} m_i \vec{r}_i = m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_n \vec{r}_n$.

$\sum_{i=1}^{n} m_i = m_1 + m_2 + \dots + m_n = M$.

$$\vec{R} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M} \quad \text{--- (II)}$$

Let x_i, y_i, z_i be the Coordinates of the point, where ith particles located.

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$$

llly X, Y and Z be the Co-ordinate of Centre of mass.

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

equation (11) becomes.

$$\hat{i}X + \hat{j}Y + \hat{k}Z = \frac{1}{M}\sum_{i=1}^{i=n} m_i(\hat{i}x_i + \hat{j}y_i + \hat{k}z_i)$$

equating the x, y, z.

$$X = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

$$Y = \frac{1}{M}\sum_{i=1}^{n} m_i y_i$$

$$Z = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

if the spacing between the particle is small,

Same ie $m = \Delta m$

$$X = \frac{1}{M}\sum_{i=1}^{n} (\Delta m_i) x_i$$

$$Y = \frac{1}{M}\sum_{i=1}^{n}(\Delta m_i)y_i$$

and $$Z = \frac{1}{M}\sum_{i=1}^{n}(\Delta m_i)Z_i \quad \text{--- (III)}$$

as n bigger and each Δm_i smaller than

$$\sum(\Delta m)x_i = \int x\,dm$$

$$\sum(\Delta m)y_i = \int y\,dm$$

and $$\sum(\Delta m)z_i = \int y\,dm$$

equation (III) becomes

$$\left.\begin{aligned} X &= \frac{1}{M}\int x\,dm \\ Y &= \frac{1}{M}\int y\,dm \\ Z &= \frac{1}{M}\int z\,dm \end{aligned}\right\} \quad \text{(IV)}$$

The vector expression

$$\vec{R} = \frac{1}{M}\int \vec{r}\,dm.$$

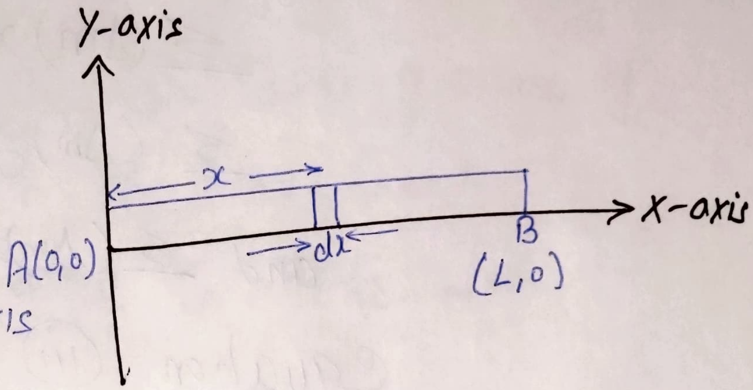
if centre of mass be on origin.

$$R(x,y,z) = 0$$

$$\frac{1}{M}\int \vec{r}\,dm = 0$$

$\left[\text{ie } \int x\,dm = \int y\,dm = \int z\,dm = 0\right]$

✱ Centre of Mass of a Uniform Thin Rod:-



Let us consider a uniform thin rod AB of Mass (M) and length (L)

The rod is held along X-axis with its A on origin (0,0).

Consider a small element (dx) at distance (x) from origin.

The mass of element dx

$$dm = \frac{M}{L}\,dx$$

Then, Co-ordinates of Centre of mass of thin rod.

$$x_{CM} = \frac{1}{M}\int_0^L x\,dm = \frac{1}{M}\int_0^L x\,\frac{M}{L}\,dx$$

$$= \frac{1}{\cancel{M}} \times \frac{\cancel{M}}{L}\int_0^L x\,dx$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{1}{L \times 2}\left[L^2 - 0\right]$$

$$x_{CM} = \frac{L^{\cancel{2}}}{2\cancel{L}} = \frac{L}{2}$$

$$x_{CM} = \frac{L}{2}$$

and $y_{CM} = \frac{1}{M}\int y\,dm$

$$y_{CM} = 0$$

Thus, it means that the Centre of mass of thin rod AB be on the Point $(\frac{L}{2}, 0)$.

Thus the Centre of mass coincides with geometric Centre.

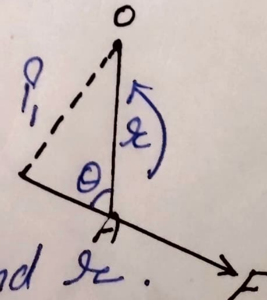
★ Moment of Force or Torque:-

The moment of force or Torque due to a force gives us the turning effect of the force about the fixed Point. It is denoted by (τ)

Momentum of force = force × ⊥ distance

$$\vec{\tau} = \vec{F} \times \vec{r}$$

$$\tau = F r \sin\theta \, \hat{n}$$



Here θ is the angle between F and r.

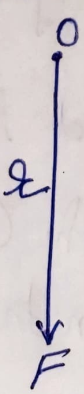
The direction of $\vec{\tau}$ is $(\perp)$ to the plane containing $\vec{F}$ and $\vec{r}$.

S.I. Unit of Torque :- N-m

dimension formula = $[ML^2T^{-2}]$.

Case I :- When $\theta = 0$

The direction of force and distance lie on same line



$\tau = rF\sin\theta$

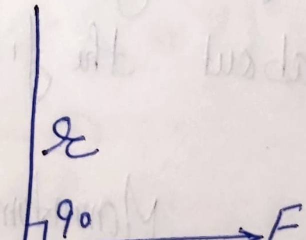
$= rF\sin(0)$

$\boxed{\tau = 0}$.

No turning effect.

Case II :- When $\theta = 90$

The direction of force and distance at right angle



$\tau = rF\sin\theta$

$= rF\sin(90)$

$\boxed{\tau = rF}$

Maximum Torque.

Case III :- $\theta = 180$

When Force and ~~distance~~ distance opposite in direction

$\tau = Fr \sin 180$

$\boxed{\tau = 0}$

Minimum Torque.

★ Angular Momentum of a Particle :-

Angular momentum (L) can be defined as moment of linear momentum about a point.

The angular momentum of a particle of mass (m) moving with velocity (v) about a point O

$\vec{L} = \vec{r} \times \vec{P}$ $\quad (\vec{P} = m\vec{v})$

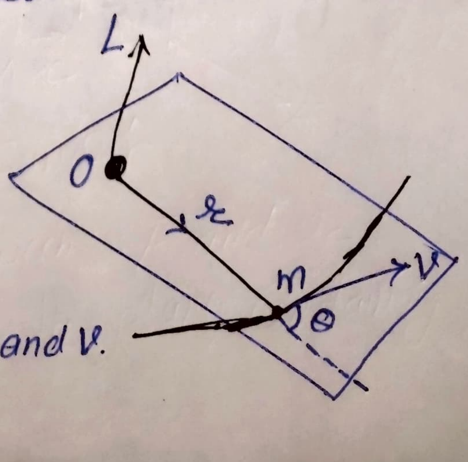
$\vec{L} = (\vec{r} \times \vec{v}) m$ — ①

L = angular momentum about point O
r = is position vector.

equation ① becomes

$L = mvr \sin\theta$

Here θ = angle between r and v.



also $p = mv$

ie $L = r p \sin\theta$

$$\boxed{L = p r_{\perp} \sin\theta}$$

where $r_{\perp}$ is $\perp$ distance of the linear momentum vector (p)

Case :-

I :- Angular momentum will be zero if $p = 0$, $r = 0$ or $\theta = 0$ or 180.

II Angular momentum is a vector quantity and its direction could be found out with the help of cross-product.

III S.I. Unit of angular momentum

$= mvr = kg(m/s)\,m$

$= kg\,m^2/s$

dimension formula :- $[ML^2T^{-1}]$

★ law of Conservation of Angular momentum and It's Applications:-

According to this principle, when no external torque acts on a system of particles, then the total angular momentum of the system remains always constant.

Total torque acting on the body due to external force

$$\vec{\tau}_{total} = \frac{d}{dt}\left[\vec{L}_{total}\right] \quad — (1)$$

$$\left[\because L = \frac{d(mvr)}{dt} = m\frac{dv}{dt}r = ma \times r \qquad \vec{\tau} = \vec{F}\times\vec{r}\right.$$

∴ when no external torque acts on the system $\vec{\tau}_{total} = 0$

∴ equation (1) becomes

$$\frac{d}{dt}\left[\vec{L}_{total}\right] = 0$$

$$\vec{L}_{total} = \text{Constant}$$

ie $\left[\boxed{\vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots + \vec{L}_n = \text{Constant}}\right.$

This is the principle of Conservation of angular momentum.

Sone Examples of Conservation of Angular Momentum

(a) **Ice Skater:** When an ice skater spins with her arms outstretched, she spins at a certain rate. But when she pulls her arms close to her body, she spins faster. When the arms are outstretched, the moment of inertia is large and the angular speed is slow. When the arms are pulled in, the moment of inertia decreases, but to conserve angular momentum, the angular speed increases.

(b) **Planetary Orbits:** A planet moves faster when it is closer to the sun. This can be explained by conservation of angular momentum. When the planet is closer to the sun, its moment of inertia decreases and its angular speed increases to compensate.

(c) **Swivel Chair with Weights:** If you sit on a swivel chair holding weights in both hands and someone gives you a push, you'll spin around. If you pull the weights close to your body, you'll start to spin faster. This is similar to the ice skater example but can be easily demonstrated in a classroom setting.

(d) **Bicycle Wheel:** Hold a bicycle wheel by its axle while it's spinning. If you try to tilt the wheel, you'll feel a resistance. This effect is due to the conservation of angular momentum and is known as gyroscopic precession.

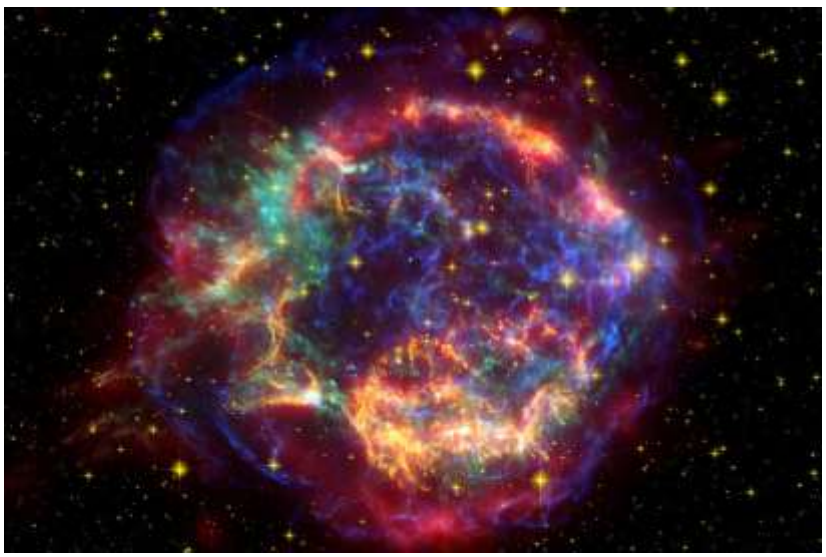
(e) **Hovercraft Rotation:** Some educational setups use a hovercraft (basically a platform on a cushion of air) that can spin freely. If a person on the hovercraft holds spinning wheels and then tilts them, the hovercraft will start rotating due to the change in angular momentum from the wheels.

(f) **Helicopter: -** All helicopters are provided with two propellers. If there were one single propeller, the helicopter would rotate itself in opposite direction in accordance with the law of conservation of angular momentum.

(g) Falling Cat: - While falling, a cat stretches its body along with its tail so that its moment of inertia (I) increases. As no external torque is acting.

$L = I\,\omega$ = constant. Since (I) increases, (ω) decreases and the cat lands gently on its feet.

(h) Dying Star: A star that is collapsing into a neutron star or a black hole spin faster as it gets smaller. The star's moment of inertia decreases as it shrinks, but its angular momentum is conserved, so its angular speed increases.



★ Equilibrium of a Rigid Body :-

A Rigid body is said to be in equilibrium if both its linear momentum angular momentum are not changing with time.

1) **Translational Equilibrium** :- A rigid body is said to be in translational equilibrium, if it remains at rest or moving with a constant velocity in a perticular direction.

that means external force is zero

$$\vec{F}_{ext} = 0$$

$$\text{ie } \sum_i \vec{F}_i = 0$$

$$\Rightarrow \sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0$$

from Newtons seconds law

$$\vec{F} = \frac{d\vec{P}}{dt}$$

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = 0$$

$$\vec{F} = \frac{m\,d(v)}{dt} = 0 \qquad (m \neq 0$$

$$\text{ie } \frac{dv}{dt} = 0 \Rightarrow \quad \text{(Integrating both side)}$$

$$\boxed{V = \text{Constant}} \text{ or zero.}$$

ie either velocity is constant or zero.

Static equilibrium:- If the body at Rest.

dynamic equilibrium:- If the body is in uniform motion.

Types of Static Equilibrium :-

1) **Stable equilibrium** :- When the body tries to regain its equilibrium position after slightly displaced and released.

eg:- table/chair lying on ground is in stable equilibrium.

2) **Unstable Equilibrium** :- When the body gets disturbed further after being slightly displaced and released.

eg:- A book standing on an edge is in unstable equilibrium.

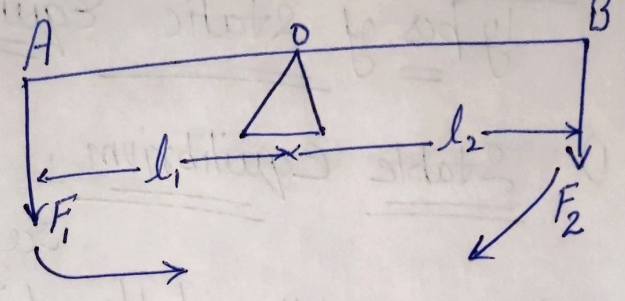
3) **Neutral Equilibrium** :- When the body can stay in equilibrium even after being slightly displaced, and released.

eg:- A ball on the ground is in neutral equilibrium.

2) Secondary Equilibrium / Rotational Equilibrium :-

A Rigid body is said to be in Rotational equilibrium, if the body does not rotate or rotates with constant angular velocity.

Consider a beam balance, the system will be in Rotational equilibrium if



$$F_1 l_1 - F_2 \times l_2 = 0$$

ie. $+\vec{\tau}_1 = F_1 l_1$ (anticlockwise)

$-\vec{\tau}_2 = F_2 \times l_2$ (Clockwise)

$$\vec{\tau}_1 + \vec{\tau}_2 = 0$$

or $\sum \vec{\tau} = 0$

i.e. for Rotational equilibrium total external torque acting on the body must be zero.

$$\vec{\tau}_{ext} = \sum_{i=1}^{i=n} \vec{\tau}_i = \frac{d\vec{L}}{dt} = 0$$

$$\frac{d\vec{L}}{dt} = 0$$

Integrating both side

$$\boxed{\vec{L} = \text{Constant}}$$

ie angular momentum stay Constant $\quad \vec{\tau}_{ext} = I\vec{\alpha} = 0$

$\therefore \vec{\alpha} = 0$

[angular accelleration will be zero in equilibrium.]

* Rigid body Rotation and Equations of Rotational Motion :- (11)

Rigid body Rotation is a motion that occurs when a solid body moves in a circular path around something.

Three simple Relations between Rational Kinematic Variable are:

(i) $\omega = \omega_0 + \alpha t$

(ii) $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$

(iii) $\omega^2 - \omega_0^2 = 2\alpha\theta$.

(1) $\omega = \omega_0 + \alpha t$

we know that

$$\alpha = \frac{d\omega}{dt}$$

$$d\omega = \alpha dt$$

Intigrating both side

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha dt$$

$$[\omega]_{\omega_0}^{\omega} = \alpha [t]_0^t$$

$$\omega - \omega_0 = \alpha t$$

$$\boxed{\omega = \omega_0 + \alpha t} \text{ ——— } (1)$$

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

ω = angular velocity

$\omega = \frac{d\theta}{dt} \Rightarrow d\theta = \omega\, dt$

Intigrating both side at $t=0 \rightarrow t=t$

$\theta = 0 \rightarrow \theta = \theta$

$$\int_0^\theta d\theta = \int_0^t \omega\, dt$$

$$[\theta]_0^\theta = \int_0^t (\omega_0 + \alpha t)\, dt$$

$$\theta = \left[\omega_0 t + \alpha \frac{t^2}{2}\right]_0^t$$

$$\boxed{\theta = \omega_0 t + \frac{1}{2}\alpha t^2}$$

(iii) $\omega^2 - \omega_0^2 = 2\alpha\theta$

We know that

$\omega = \frac{d\theta}{dt}$ and $\alpha = \frac{d\omega}{dt}$

$\alpha = \frac{d}{dt}(\ldots) \Rightarrow \alpha = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt}$

$\alpha = \omega \frac{d\omega}{d\theta} \Rightarrow \alpha\, d\theta = \omega\, d\omega$

Integrating both side

$\theta = 0 \rightarrow \theta = \theta$

$\omega = \omega_0 \rightarrow \omega = \omega$

$$\int_0^\theta \alpha \, d\theta = \int_{\omega_0}^{\omega} \omega \, d\omega$$

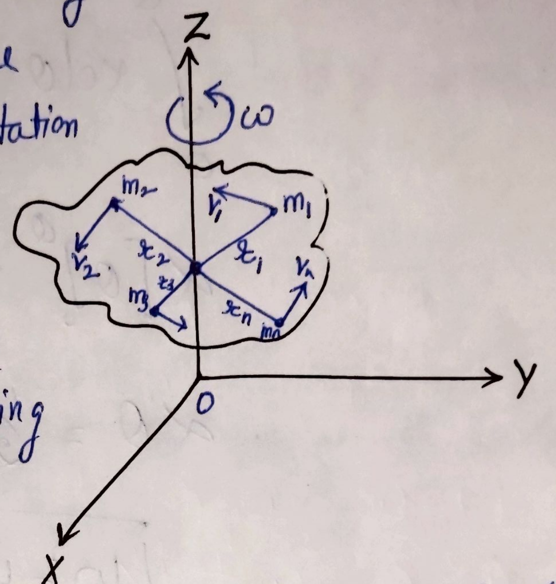
$$\alpha [\theta]_0^\theta = \left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega}$$

$$\alpha \theta = \frac{1}{2}[\omega^2 - \omega_0^2]$$

$$\Rightarrow \boxed{\omega^2 - \omega_0^2 = 2\alpha\theta}$$

★ Kinetic Energy and Momentum of Inertia :-

Kinetic energy of rotating of a body is the energy possessed by the body on account of its rotation about a given axis.



Let us Conside a Rigid body having n- number of particles rotating around a z-axis in xy plane.

Let $m_1, m_2 \ldots m_n$ be the masses and $r_1, r_2 \ldots r_n$ be the respective Perpendicular distance from axis of rotation.

The angular velocity of each particle is same But linear velocity is $V_1, V_2, V_3 \ldots V_n$.

ie $V_1 = r_1\omega$, $V_2 = r_2\omega$, --- $V_n = r_n\omega$

Kinetic energy of particle of mass m_1

$$K.E = \frac{1}{2}m_1V_1^2 = \frac{1}{2}m_1(r_1\omega)^2 = \frac{1}{2}m_1r_1^2\omega^2$$

||^{ly} another particle

$$\frac{1}{2}m_2V_2^2 = \frac{1}{2}m_2r_2^2\omega^2 \quad \ldots \quad \frac{1}{2}m_nr_n^2\omega^2$$

Total Kinetic energy of the Rigid body

$$K.E. = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \ldots + \frac{1}{2}m_nr_n^2\omega^2$$

$$K.E = \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2)\omega^2$$

$$K.E. = \frac{1}{2}\left[\sum_{i=1}^{n} m_i r_i^2\right]\omega^2$$

Here $\boxed{I = \sum_{i=1}^{n} m_i r_i^2}$ = moment of Inertia.

$$\boxed{K.E. = \frac{1}{2} I \omega^2}$$

Moment of Inertia :- It is the sum of the product of the masses of all the particles of the body and squares of their respective perpendicular distances from the axis of rotation.

$$I = \sum_{i=1}^{n} m_i r_i^2$$

It depends upon

(I) Position of the axis of Rotation
(II) Orientation of the axis of Rotation
(III) Shape of the body
(IV) Size of the body
(V) distribution of mass of the body about the axis of rotation.

S.I Units and dimension formula :-

$$I = m \times (\text{distance})^2$$

$m \to$ kg, $(\text{distance})^2 \to m^2$

S.I = $kg\,m^2$

dimension formula :- $[ML^2T^0]$

Note :- when the angular velocity of the body is Unity ie $\omega = 1$

$$K.E = \frac{1}{2} I \omega^2$$

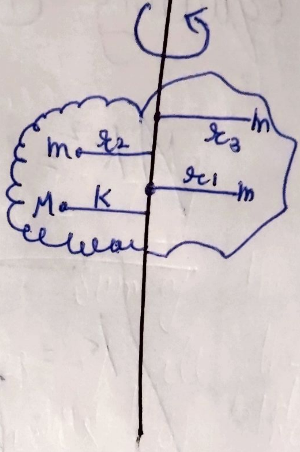
$$\boxed{I = 2K.E}$$

moment of Inertia is twice of the kinetic energy.

$$\boxed{I = 2 \times K.E \text{ of Rotation}}$$

* Radius of Gyration :-

The Radius of gyration of a body about an axis may be define as the distance from the axis of a mass point whose mass is equal to the mass of whole body and moment of inertia also equal to the moment of inertia of the body.



Let us consider a body having n number of particles of mass (m) and perpendicular distance $r_1, r_2, \dots r_n$.

Here the moment of inertia of the body

$$I = mr_1^2 + mr_2^2 + \dots + mr_n^2$$

$$= m(r_1^2 + r_2^2 + \dots + r_n^2)$$

$$= m \times n\left(\frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}\right)$$

$$I = M\left[\frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}\right] \quad (1) \quad [\because M = mn]$$

M = total mass of the body.

k = ⊥ distance from axis of Rotation

$$I = Mk^2 \text{ ——— (II)}$$

equating (I) and (II)

$$\cancel{M}k^2 = \frac{\cancel{M}(r_1^2 + r_2^2 + \cdots + r_n^2)}{n}$$

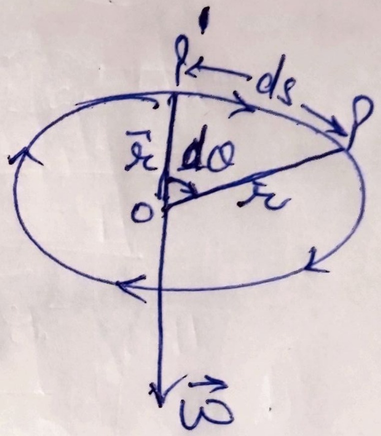
$$\boxed{k = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}}}$$

Radius of gyration of a body about a given axis is equal to the root mean square distance of the constituent particles of the body from the given axis.

Some Important Relations

① Relation between Angular velocity and Linear Velocity :-

Linear velocity :-



$$v = \frac{ds}{dt} \quad \text{—} (1)$$

$$\text{angle} = \frac{\text{arc}}{\text{Radius}}$$

$$d\theta = \frac{ds}{r} \Rightarrow ds = r\,d\theta \quad \text{—} (11)$$

Put equation (11) in (1)

$$v = \frac{r\,d\theta}{dt} \qquad \left[\frac{d\theta}{dt} = \omega \right)$$

$$v = r\omega$$

$$\boxed{\vec{v} = \vec{r} \times \vec{\omega}}$$

② Relation between Angular acceleration and Linear acceleration :-

we know that

$$v = \omega r$$

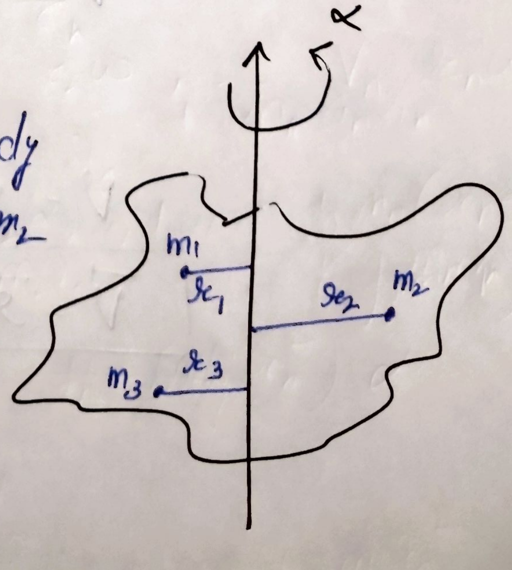
differentiating both side w.r.t. t

$$\frac{dv}{dt} = r \frac{d\omega}{dt}$$

$$\boxed{a = r\alpha}$$

$$\boxed{\vec{a} = \vec{r} \times \vec{\alpha}}$$

③ Relation between Torque and Moment of Inertia :-



let us consider a Rigid of body of n-particle of mass m_1, m_2 .. m_n and ⊥ distance from axis of Rotation $r_1, r_2 .. r_n$

α = angular acceleration [angular acc. same for all
a = linear acceleration [particle but diff. linear acc.
i.e. $a_1, a_2 \ldots a_n$]

we know that

$$a_1 = r_1\alpha, \quad a_2 = r_2\alpha \quad \ldots \quad a_n = r_n\alpha \quad \text{(1)}$$

Now force on the particles of $m_1, m_2 \ldots m_n$

$$f_1 = m_1 a_1, \quad f_2 = m_2 a_2 \quad \ldots \quad f_n = m_n a_n$$

This force produce torque in body

$$\tau_1 = f_1 \times r_1 = m_1 a_1 r_1$$

$$\tau_2 = f_2 \times r_2 = m_2 a_2 r_2$$

$$\vdots$$

$$\tau_n = f_n \times r_n = m_n a_n r_n$$

Total torque or moment of force

$$\tau = \tau_1 + \tau_2 \ldots + \tau_n$$

$$\tau = m_1 a_1 r_1 + m_2 a_2 r_2 + \ldots + m_n a_n r_n$$

$$= m_1(r_1\alpha)r_1 + m_2(r_2\alpha)r_2 + \ldots + m_n(r_n\alpha)r_n$$

[from (1)]

$$= (m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2)\alpha$$

and we know that

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2$$

$$\boxed{\tau = I\alpha}$$ $\left[I = \sum_{i=1}^{i=n} m_i r_i^2 \right]$

In vector form

$$\boxed{\vec{\tau} = I\vec{\alpha}}$$

(4) Angular Momentum and Moment of Inertia

Let us consider a rigid of n-particles of mass $m_1, m_2 \cdots m_n$ at $\perp$ distance $r_1, r_2 \cdots r_n$ respectively

Angular velocity (ω) is same for all the particle But linear velocity is different ie $V_1, V_2 \cdots V_n$.

$$V_1 = r_1\omega, \; V_2 = r_2\omega, \; \cdots \; V_n = r_n\omega.$$

linear momentum of particle

$$P_1 = m_1 V_1, \; P_2 = m_2 V_2 \; \cdots, \; P_n = m_n V_n$$

~~Total momentum of the body~~.

$P_1 = m_1(r_1\omega)$, $P_2 = m_2(r_2\omega)$ -- $P_n = m_n(r_n\omega)$

Angular momentum

$$L_1 = P_1 r_1 = m_1(r_1\omega)r_1 = m_1 r_1^2 \omega$$

||rly $L_2 = m_2 r_2^2 \omega$ --- $L_n = m_n r_n^2 \omega$

$\therefore$ Angular momentum of the Rigid body

$$L = L_1 + L_2 + L_3 - - - + L_n$$

$$L = m_1 r_1^2 \omega + \cancel{A}\, m_2 r_2^2 \omega + - - - + m_n r_n^2 \omega$$

$$L = (m_1 r_1^2 + m_2 r_2^2 + - - + m_n r_n^2)\omega$$

$$L = I\omega \qquad \left[\because I = \sum_{i=1}^{n} m_i r_i^2\right]$$

In vector form

$$\boxed{\vec{L} = I\vec{\omega}}$$

(5) Relation between Torque and Angular momentum :-

we know that

$$L = I\omega$$

differentiating both side w.r.t. t

$$\frac{dL}{dt} = I\left(\frac{d\omega}{dt}\right)$$

$$\frac{dL}{dt} = I\alpha \quad \text{— (I)}$$

Here $\boxed{\tau = I\alpha}$ — (II)

from (I) and (II)

$$\boxed{\frac{dL}{dt} = \tau}$$

Comparison of Linear and Rotational motions

	Parameter	Linear Motion	Rotational Motion
1.	**Description**	Motion along a straight line.	Motion about a fixed axis or point.
2.	**Position**	Described by linear displacement.	Described by angular displacement (θ).
3.	**Velocity**	Linear velocity (v).	Angular velocity (ω).
4.	**Acceleration**	Linear acceleration (a).	Angular acceleration (α).
5.	**Mass**	Mass (m).	Moment of inertia (I).
6.	**Force**	Force (F).	Torque (τ).
7.	**Newton's 2nd Law**	$F = ma$	$\tau = I\alpha$
8.	**Kinetic Energy**	$^1/_2\, mv^2$	$^1/_2\, I\omega^2$
9.	**Momentum**	Linear momentum (p) = mv.	Angular momentum (L) = $I\omega$.
10.	**Conservation Law**	Conservation of linear momentum.	Conservation of angular momentum.
11.	**Work Done**	$W=F\times d$ (where d is displacement).	$W = \tau \times \theta$
12.	**Power**	$P=F\times v$.	$P = \tau \times \omega$
13.	**Impulse**	Impulse = $F \times \Delta t$	Angular impulse = $\tau \times \Delta t$

Values of moments of Inertia for Simple Geometrical objects

S.no.	Body	Axis	Moment of Inertia
1.	**A uniform rod of length (L)**	perpendicular to rod through its centre	${}^{1}/_{2}\,ML^2$
2.	**Uniform rectangular lamina of length (l) and breadth (b)**	perpendicular to lamina and through its centre	$M\left(\frac{l^2+b^2}{12}\right)$
3.	**Uniform circular ring of radius R**	perpendicular to its plane and through the centre	MR^2
4.	**Uniform circular ring of radius R**	Diameter	$\frac{1}{2}MR^2$
5.	**Uniform circular disc of radius R**	perpendicular to its plane and through the centre	$\frac{1}{2}MR^2$
6.	**Uniform circular disc of radius R**	Diameter	$\frac{1}{4}MR^2$
7.	**Hollow cylinder of radius R**	Axis of the cylinder	MR^2
8.	**Solid cylinder of radius R**	Axis of the cylinder	$\frac{1}{2}MR^2$
9.	**Hollow sphere of radius R**	Diameter	$\frac{2}{3}MR^2$
10.	**Solid sphere of radius R**	Diameter	$\frac{2}{5}MR^2$